Evaluating Tolerance Interval Estimates

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Comparing Interval Estimates

- Statistical interval estimates are constructed to
  - Estimate parameters
  - Quantify characteristics of population
- To correctly interpret estimates, it must be clearly defined what each interval is estimating
  - Confidence/prediction intervals are well understood
  - Definition of a tolerance interval varies among literature sources

What is a Tolerance Interval?

- Tolerance intervals are being used with more frequency, thus a consistent definition needs to be established
- Definitions found in literature:
  - A bound that covers at least (100-α)% of the measurements with (100- γ)% confidence (*Walpole and Myers)*
    - Focuses on where individual observations fall
    - Equivalent to a (100-γ)% CI on middle (100-α)% of Normal distribution
  - An interval that includes a certain percentage of measurements with a known probability (*Mendenhall and Sincich)*
    - TI is identical to CI, except it attempts to capture a proportion of measurements rather than a population parameter, such as μ
- Computational methods vary depending on author
  - *Mee's* definition uses non-central t distribution
  - *Owen et al.* propose to control the percentage in both tails of a distribution
    - No more than a specified proportion lies below or above the TI
- Alternative definition gives interval estimates on lower and upper *percentiles*, not a *percentage*, of a distribution

$$\text{lower TI} = [\bar{X} - (\alpha/2,n\text{-}1,\delta))\frac{s}{\sqrt{n}},\ \bar{X} - t(1\text{-}\alpha/2,n\text{-}1,\delta)\frac{s}{\sqrt{n}}] \text{ and}$$

$$\text{upper TI} = [\bar{X} + t(\alpha/2,n\text{-}1,\delta))\frac{s}{\sqrt{n}},\ \bar{X} + t(1\text{-}\alpha/2,n\text{-}1,\delta)\frac{s}{\sqrt{n}}], \text{ where } \delta = \Phi(\text{percentile})\sqrt{n}$$

Other Interval Estimates

- *Simultaneous tolerance interval:*
  - Tolerance interval computed for more than one population or sample at a time
- *Two 1-sided tolerance interval:*
  - 1-sided TI on lower α/2% together with 1-sided TI on upper α/2%, designed to capture (1-α)% of distribution
- *Confidence/prediction interval on confidence/prediction interval endpoints:*
  - Interval that protects the mean/future confidence/prediction interval endpoints (based on asymptotic Normal distribution of interval endpoints)
- *β-expectation tolerance interval (Mee's definition):*
  - Interval that contains approximately 100β% of the distribution:

$$E_{\hat{\mu},\hat{\sigma}_x}\{Pr_X[\hat{\mu} - k\hat{\sigma}_x < X < \hat{\mu} + k\hat{\sigma}_x | \hat{\mu},\hat{\sigma}_x]\} = \beta$$

$$\sigma_x^2 = \sigma_b^2 + \sigma_e^2 \qquad \hat{\sigma}_x^2 = MSB/J + (1\text{-}1/J)MSE$$

I = # of batches, runs, blocks, etc.   J = # of reps within runs

  - Computed using σ_b^2/σ_e^2 (ratio of between to within batch variance) and central *t*-distribution
  - Variance is a linear combination of independent mean squares, df calculated using Satterthwaite approximation
- *β-content tolerance interval:*

$$Pr_{\hat{\mu},\hat{\sigma}_x}\{Pr_X[\hat{\mu} - k\hat{\sigma}_x < X < \hat{\mu} + k\hat{\sigma}_x | \hat{\mu},\hat{\sigma}_x] \geq \beta\} = \gamma$$

γ = confidence coefficient

  - Interval that contains at least 100β% of population with given confidence level γ (*Mee*)
    - Computed using factors from Normal and Chi-squared distributions
    - β-content interval in models with only 1 source of variation are computed using noncentral *t*-distribution
  - *Wald and Wolfowitz* use same definition to define tolerance intervals but instead use the formula:

$$\bar{X} \pm \sqrt{\frac{n}{\chi^2_{n,\beta}}}\, rs$$

- SAS® Proc Capabilities Method 3 computes an approximate statistical tolerance interval that contains at least *p* proportion of the population with formula given by:

$$\bar{X} \pm z_{\frac{1+p}{2}}(1+1/2n)\sqrt{\frac{n-1}{\chi^2_{\alpha(n-1)}}}\, s$$

- *Patel* states 1-sided tolerance limits are directly related to 1-sided confidence limits on percentiles
  - Formulas are provided for constructing TI when one or both μ and σ are unknown

Evaluation of Interval Estimates

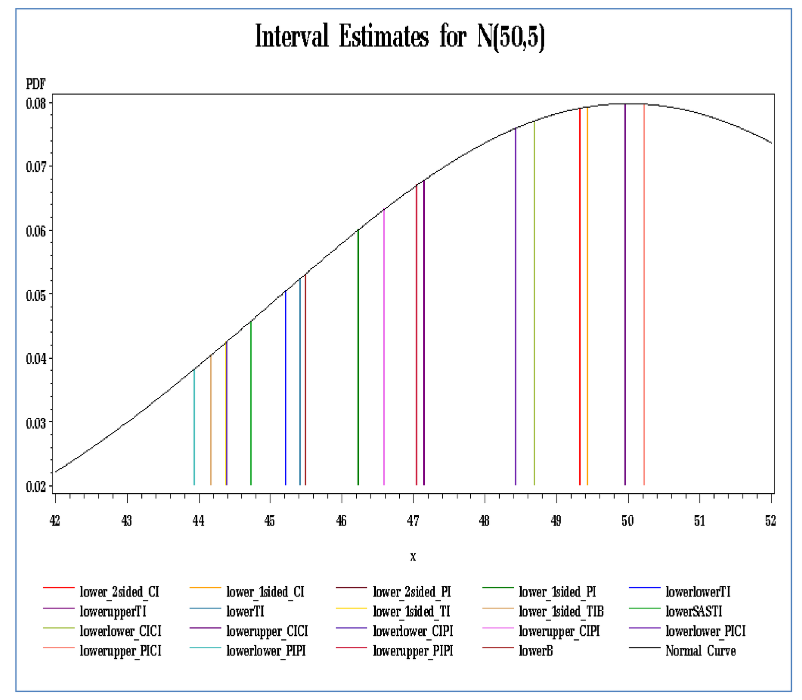
- The relationship among prediction, β-expectation, and β-content intervals is investigated
  - Prediction intervals are compared with β-expectation intervals for models with 1 variance component (all factors fixed)
  - β-expectation intervals are compared with β-content intervals

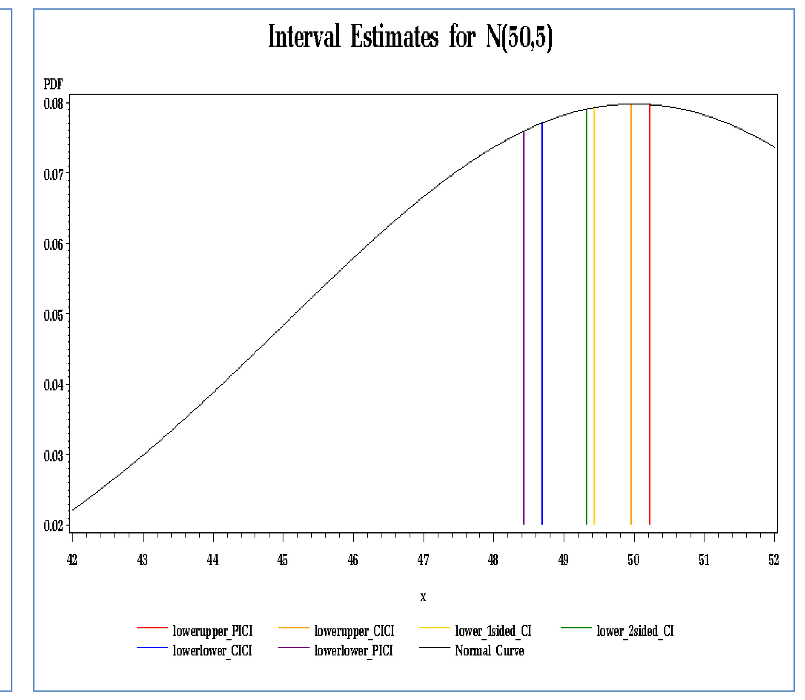
Prediction vs. β-Expectation Tolerance Intervals

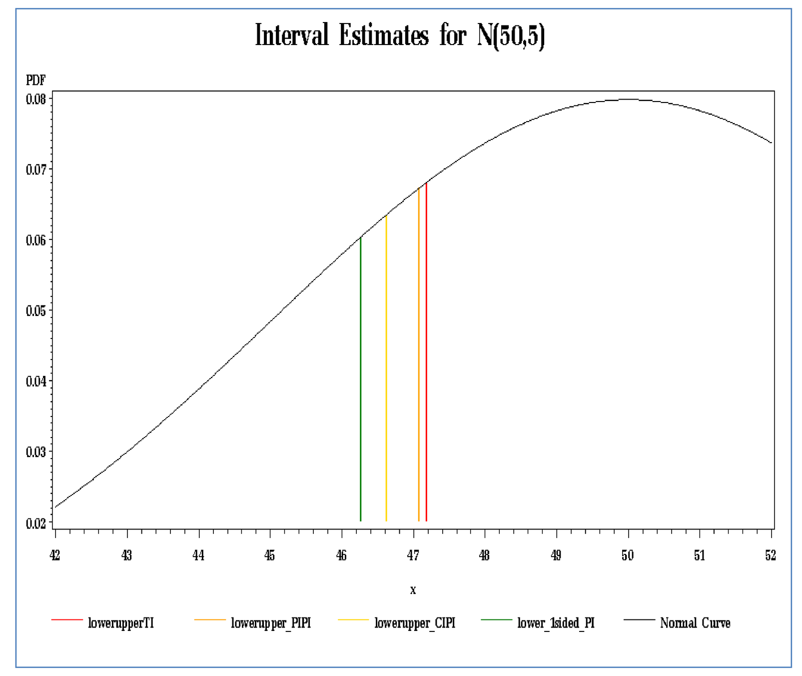
- 1 variance component
  - β-expectation interval equals prediction interval for both overall mean and treatment means (regardless of the number of factors in model)
    - Here $\sigma_b^2/\sigma_e^2 = 0$ because = 0
    - No df adjustment needed
  - Formula for β-expectation interval reduces to formula for prediction interval

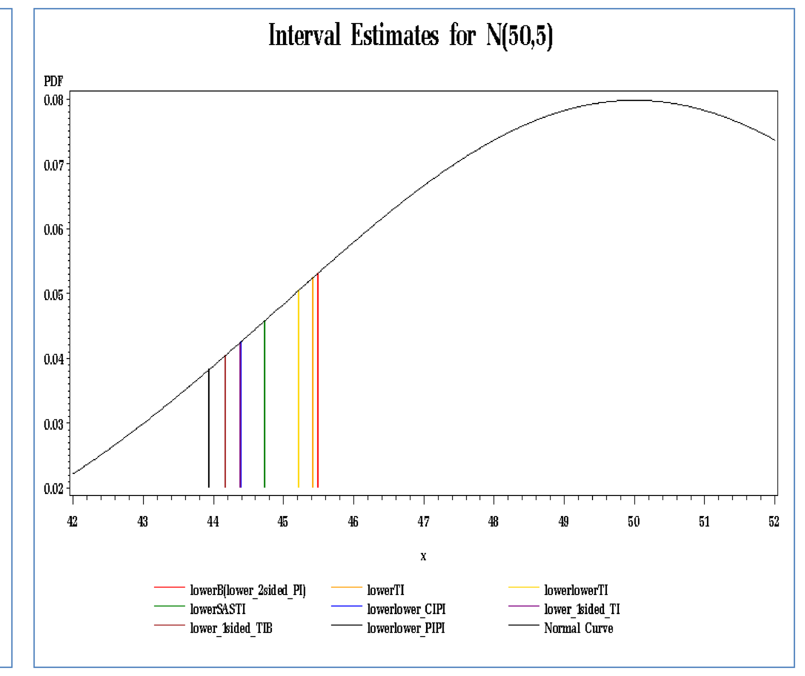
Comparing Interval Estimates

- To understand tolerance intervals and their relationship among other interval estimates for one sample with one variance component, a computer simulation was conducted
  - 50 iterations with data sets of size 50
- Observations were randomly generated, various interval estimates were constructed and compared
- The mean of each interval estimate across iterations is computed
- Comparisons are made among the interval estimates









Characteristics of Interval Estimates

- Interval estimates can be characterized into 3 groups:
  - Group 1: Bounds on mean or on a CI endpoint
    - lowerupper PICI, lowerupper CICI, lower 1-sided CI, lower 2-sided CI, lowerlower CICI, lowerlower PICI
  - Group 2: Bounds on individual observations, PI endpoints, or upper bound on lower α/2 percentile
    - lowerupper TI, lowerupper PIPI, lowerupper CIPI, lower 1 sided PI
  - Group 3: Combination of bounds on individual observations, PI endpoints, or lower bound on lower α/2 percentile
    - lower β-expectation (i.e. lower 2-sided PI), 2 1-sided lower TI, lowerlower TI, lower SAS TI, lowerlower CIPI, lower 1-sided TI, lower 1 sided TI with Bonferroni correction, lowerlower PIPI

Conclusions

- Current literature sources offer a wide variety of definitions for tolerance intervals
- Prediction intervals equal β-expectation tolerance intervals for models with 1 variance component
- β-content intervals TI are wider than β-expectation TI
- 1-sided TI is directly related to a 2-sided TI on percentile
- Simulation results indicate
  - Interval estimates can be characterized into 3 groups
  - It must be determined the correct parameter of interest to be captured to determine which estimate to use